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Lower Upper Implicit Total Variation Diminishing Solution of Viscous Hypersonic Flows

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Abstract

IN this paper an efficient implicit LU-TVD scheme is developed to solve the two-dimensional Navier-Stokes equations. The scheme uses a lower and upper approximate factorization of the implicit operator and has been applied to a viscous high-speed flow problem for validation purposes. Extensive testing indicates that the efficiency of a TVD method can be greatly increased without losing accuracy.

Contents

The solution of the full Navier-Stokes equations requires a strong computational effort (CPU time, storage capacity, etc.). For this reason, an implicit efficient high-order scheme has been developed and applied to solve the Navier-Stokes equations for high-speed flows. Among the implicit schemes, attention has been focused on the ones that use approximate factorization of the solution operator.

Recently, Jameson and Yoon[1] applied the LU scheme to solve transonic and supersonic flows for the Euler equations cast in a finite volume formulation. They also developed a modified LU scheme, called LU-SSOR, that uses a symmetric successive over-relaxation technique, by splitting the inviscid flux Jacobians in such a way to increase the diagonal dominance of the LU factors and to avoid matrix inversion. Lately, Rieger and Jameson[2] extended the LU-SSOR scheme to three dimensions to solve the steady compressible Navier-Stokes equations in the framework of a finite volume discretization technique.

In the present work a finite volume implicit solver is developed to solve the laminar Navier-Stokes equations for hypersonic flows. The numerical algorithm is based on a symmetric discretization of the viscous terms, and it uses an upwind biased second-order TVD scheme for the inviscid terms.[3,4] The performance of the method is improved by using an implicit approach with a lower and upper factorization of the implicit operator.

The governing equations in conservation form are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{W}\,\mathrm{d}S = -\oint_{\partial S} (\boldsymbol{F}_E - \boldsymbol{F}_V)\cdot\boldsymbol{n}\,\mathrm{d}s \tag{1}$$

where $\boldsymbol{W}$, $\boldsymbol{F}_E$, and $\boldsymbol{F}_V$ are, respectively, the unknown vector and the inviscid and viscous flux tensors.[3]

A cell-centered finite volume approach is employed, whereby the computational domain is decomposed into arbitrary quadrilateral cells (i,j) whose surface is $S_{i,j}$. Space and time discretization are separated by using the method of lines, thus reducing Eq. (1) to a system of ordinary differential equations.

Linearization of the flux vectors about the previous time step yields an unfactored implicit scheme. Let A and $\boldsymbol{P}$ be, respectively, the Jacobians of the inviscid and viscous fluxes in the direction normal to cell faces $\beta = 1,3$:

$$A = \left[\frac{\partial}{\partial \boldsymbol{W}}(\boldsymbol{F}_E^x\, n_x + \boldsymbol{F}_E^y\, n_y)\right]_{\beta=1,3}$$

$$\boldsymbol{P} = \left[\frac{\partial}{\partial \boldsymbol{W}}(\boldsymbol{F}_V^x\, n_x + \boldsymbol{F}_V^y\, n_y)\right]_{\beta=1,3}$$

Likewise $\boldsymbol{B}$ and $\boldsymbol{Q}$ represent the inviscid and viscous flux Jacobians at cell faces $\beta = 2,4$. Then, by means of the mean value theorem and midpoint rule, the following discretized form of the equations is obtained:

$$S_{i,j}\frac{\Delta \boldsymbol{W}_{i,j}}{\Delta t} + \eta\sum_{\beta=1,3}[(A+\boldsymbol{P})\Delta s\Delta \boldsymbol{W}]_\beta + \eta\sum_{\beta=2,4}[(\boldsymbol{B}+\boldsymbol{Q})\Delta s\Delta \boldsymbol{W}]_\beta = \boldsymbol{R}_{i,j}^n \tag{2}$$

where $\boldsymbol{R}_{i,j}^n$ is the residual of the equation, $\Delta \boldsymbol{W}_\beta$ is the time variation of the solution at cell face β, and η is a weighting factor that weighs the numerical flux between the time levels n an $n+1$. The residual is defined as

$$\boldsymbol{R}_{i,j}^n = \sum_{\beta=1}^{4}[(\boldsymbol{F}_{E,\,\mathrm{num}} - \boldsymbol{F}_{V,\,\mathrm{num}})\cdot\boldsymbol{n}\,\Delta s]_\beta \tag{3}$$

The numerical inviscid flux $\boldsymbol{F}_{E,\,\mathrm{num}}$ is evaluated by using a second-order upwind biased total variation diminishing formulation as proposed by Yee et al.,[4] with a *minmod* limiter (its dissipative character is reduced within the boundary layer by local Mach number scaling). The numerical viscous flux is evaluated by using a symmetric discretization.

A block triangular form of the implicit operator is obtained as follows. First, using flux difference concepts, the contribution of the inviscid flux Jacobians at each cell face is split into the positive and negative part. The positive (negative) inviscid flux Jacobian matrix A^+ (A^-) is constructed so that it has nonnegative (nonpositive) eigenvalues. To obtain a well-conditioned diagonally dominant operator matrix, the following formulas[4] have been used:

$$A^{\pm} = \omega^e(A \pm \rho_A I)/2 \tag{4}$$

$$\rho_A = r_A \max(|\lambda(A)|) \tag{5}$$

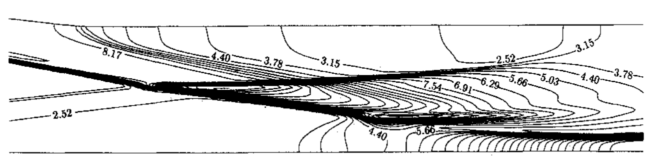

Fig. 1 Isopressure lines ($\Delta p = 0.7$), $X = 9.2$–25.3 in.

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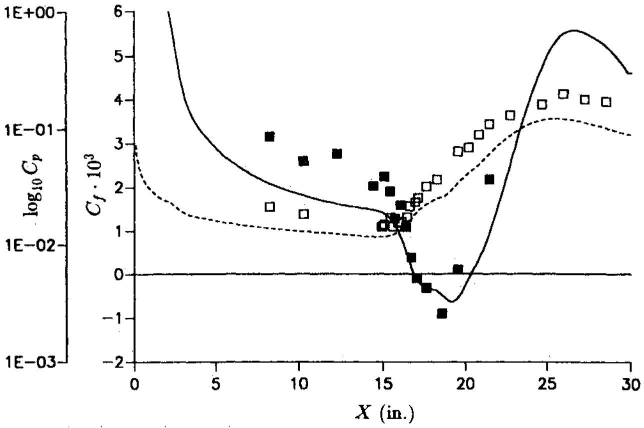

Fig. 2 Pressure coefficient (□ Ref. 6,---present method) and skin friction coefficient (■ Ref. 6,—— present method) vs X (in.).

where $\lambda(A)$ are the eigenvalues of the inviscid flux Jacobian, r_A is a constant of $\mathcal{O}(1)$ that affects the stability and convergence, ω^e is a relaxation parameter that enhances the effects of the inviscid eigenvalues (a value of ω^e equal to 2 has been used), and I is the unit matrix. Then, at cell face $i+½, j$, one has

$$(A\Delta W)_{i+½,j} = A^+_{i,j}\Delta W_{i,j} + A^-_{i+1,j}\Delta W_{i+1,j} \tag{6}$$

The viscous flux Jacobians are reduced to a diagonal form by using the spectral radii,[5] thus obtaining

$$P = \omega^v\, k_x n_x I \tag{7}$$

where k_x is the spectral radius of the viscous Jacobian matrix and ω^v is a relaxation parameter that inhibits the effects of the viscous eigenvalues (a value of ω^v equal to 1/2 has been used). Similarly for B and Q.

To reduce the computational effort, a symmetric successive over-relaxation technique is implemented,[1,2] thus eliminating the need for 4×4 matrix inversion at each cell. Substituting Eqs. (4–7) and neglecting the variations of Δs_β in the x and y directions (strictly valid only for undistorted cells), the following factorized form is obtained:

$$L \cdot U\Delta W_{i,j} = \beta N R_{i,j} \tag{8}$$

where

$$N = [1 + \alpha(\rho_A \Delta s_{1,3} + \rho_B \Delta s_{2,4})]I \tag{9}$$

$$L = N - \alpha(A^+_{i-1,j}\Delta s_3 + B^+_{i,j-1}\Delta s_4) \tag{10}$$

$$U = N + \alpha(A^-_{i+1,j}\Delta s_1 + B^-_{i,j+1}\Delta s_2) \tag{11}$$

and

$$\alpha = \eta(\Delta t/S)_{i,j}, \qquad \beta = (\Delta t/S)_{i,j}, \qquad \Delta s_{\ell,m} = \frac{\Delta s_\ell + \Delta s_m}{2}$$

For steady-state solutions, the efficiency of the method has been improved by freezing the evaluation of the Jacobian matrices for $\mathcal{O}(m)$ cycles. Numerical experiments indicate that the higher the Mach number is, the more often the Jacobian matrices need to be evaluated: for supersonic flows m is $\mathcal{O}(10)$; for hypersonic flows m is $\mathcal{O}(2)$.

The solution of Eq. (8) is obtained in three phases.

Forward Sweep

Once the residual is calculated at time level n, the forward sweep is performed:

$$L\Delta W^*_{i,j} = \beta N R^n_{i,j} \tag{12}$$

At the left and bottom boundaries ΔW^* is set equal to zero.

Backward Sweep

Having obtained $\Delta W^*_{i,j}$, the domain is swept in the backward direction, starting from the top right-most corner:

$$U\Delta W_{i,j} = \Delta W^*_{i,j} \tag{13}$$

At the top and right boundaries ΔW is set equal to zero.

Updating of the Solution

Once ΔW is computed, the solution is updated as follows:

$$W^{n+1}_{i,j} = W^n_{i,j} + \Delta W_{i,j} \tag{14}$$

Results and Discussion

The problem analyzed is the interaction of a shock, generated by a shock-generator wedge, impinging on a hypersonic boundary layer over a flat plate. The wedge angle is $\theta = 6.45$ deg; the (generated) oblique shock angle is $\beta = 10.5$ deg; the freestream Mach number is $M_\infty = 15.57$; and the Reynolds number $Re_L = 0.337\times10^6$, where $L = 30$ in. is the length of the plate. The freestream temperature is $T_\infty = 136.22$ K and the wall temperature is $T_w = 963$ K. For these nominal test case conditions the flow is fully laminar[6] and real gas effects are negligible on account of the low temperature. This test case is a severe test for the numerical scheme due to the complexity of the problem. The leading-edge shock interacts with the wedge-generated shock. The former is transmitted and interacts with the expansion occurring at the trailing edge of the wedge. The latter shock impinges on the flat plate, causing the flow to separate as observed from the isopressure contour lines reported in Fig. 1.

The mesh used has 198×150 cells with normal spacing Δy ranging from 0.5×10^{-5} to 0.18×10^{-2} and an aspect ratio that varies between 3 (near the wall) and 110. A stable solution has been obtained by using a CFL number equal to 2 and by evaluating the Jacobian matrices every five cycles. The pressure coefficient C_p and the skin friction coefficient C_f vs X are reported in Fig. 2. The computed results are in good agreement with the experiments of Holden[6]: both separation and reattachment points are well predicted, as well as the Stanton number (not reported here). Numerical results show that the present scheme is as accurate as an explicit TVD method and as efficient as an explicit adaptive dissipation (AD) one. The performance of the scheme has been assessed by comparing the CPU time (τ) per cycle per unit cell with that of the explicit TVD and AD methods. From the computed results on a scalar machine IBM RISC-6000, we find that the present scheme (with the evaluation of the Jacobian matrices every five cycles) is 40% faster than the explicit TVD method ($\tau_{LU} = 5.6\times10^{-4}$ s, $\tau_{TVD} = 9.0\times10^{-4}$ s) and 40% slower than the explicit dissipation method ($\tau_{AD} = 4.0\times10^{-4}$ s).

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